

A Randomized Version of the Collatz $3x + 1$ Problem

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Abstract

We propose a stochastic version of the Collatz $3x + 1$ Problem.

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1 The Collatz $3x + 1$ problem

The classical Collatz $3x + 1$ Problem can be formulated as follows. Let x be a positive odd integer. Consider the sequence

$$x_0 = x, \quad x_n = \frac{3x_{n-1} + 1}{2^{d_n}}, \quad n \geq 1, \quad (1.1)$$

where 2^{d_n} is the highest power of 2 dividing $3x_{n-1} + 1$. Hence $\{x_n\}_{n=0}^{\infty}$ is a sequence of positive odd integers (if, e.g., $x = 1$, then $x_n = 1$ for all n). Notice that $d_n \geq 1$ for all $n \geq 1$.

Suppose $L := \liminf x_n < \infty$. Then, since x_n takes only positive integral values, we must have that $x_n = L$ for infinitely many values of n . In particular, $x_k = x_{k+b} = L$ for some $k \geq 0$, $b \geq 1$. But, then, it follows from (1.1) that $x_{k+n} = x_{k+b+n}$ for all integers $n \geq 0$. Therefore, either

$$\lim_n x_n = \infty, \quad (1.2)$$

or the sequence $\{x_n\}_{n=0}^\infty$ is eventually periodic, namely there is a $b \geq 1$ and an $n_0 \geq 0$ such that

$$x_{n+b} = x_n \quad \text{for all } n \geq n_0. \quad (1.3)$$

Notice that, if $b = 1$, i.e. if there is a n_0 such that $x_{n+1} = x_n$ for all $n \geq n_0$, then (1.1) implies that $(2^{d_{n+1}} - 3)x_n = 1$, which forces $x_n = 1$ for all $n \geq n_0$. The Collatz $3x+1$ Problem pertains to the behavior of the sequence $\{x_n\}_{n=0}^\infty$ as $n \rightarrow \infty$. One famous and longstanding open question is whether there exists some initial value x for which $\lim_n x_n = \infty$, while another open question is whether it is possible to have an eventually periodic behavior with a (minimal) period $b > 1$.

The ultimate Collatz Conjecture is that, no matter what the initial value x is, we always have that $x_n = 1$ for all n sufficiently large. Needless to say that the conjecture has been verified for a huge set of initial values x .

2 A randomized version of the problem

Let x be a positive odd integer. We consider the sequence

$$X_0 = x, \quad X_n = \frac{3X_{n-1} + \xi_n}{2^{d_n}}, \quad n \geq 1, \quad (2.1)$$

where $\{\xi_n\}_{n=1}^\infty$ is a sequence of independent and identically distributed (i.i.d.) random variables taking odd integral values ≥ -1 , and 2^{d_n} is the highest power of 2 dividing $3X_{n-1} + \xi_n$ (notice that we again have $d_n \geq 1$ for all $n \geq 1$). Thus, $\{X_n\}_{n=0}^\infty$ is now a random sequence of positive odd integers.

Let us introduce the filtration

$$\mathcal{F}_0 := \{\emptyset, \Omega\}, \quad \mathcal{F}_n := \sigma(\xi_1, \dots, \xi_n), \quad n \geq 1, \quad (2.2)$$

where (Ω, \mathcal{F}, P) is the underlying probability space. Clearly, by (2.1) we have that the random variables X_n and d_n are \mathcal{F}_n -measurable for all $n \geq 1$. Notice that

$$\mathcal{F}_n^X := \sigma(X_1, \dots, X_n) \subset \mathcal{F}_n \quad \text{and} \quad \mathcal{F}_n^d := \sigma(d_1, \dots, d_n) \subset \mathcal{F}_n, \quad n \geq 1. \quad (2.3)$$

Of course,

$$\mathcal{F}_n^X \vee \mathcal{F}_n^d = \mathcal{F}_n, \quad n \geq 1, \quad (2.4)$$

where $\mathcal{F}_n^X \vee \mathcal{F}_n^d$ denotes the σ -algebra generated by \mathcal{F}_n^X and \mathcal{F}_n^d .

Formula (2.1) implies that $\{X_n\}_{n=0}^\infty$ is a Markov chain with respect to \mathcal{F}_n , whose state space is the set \mathbb{N}_{odd} of positive odd integers. Actually, the two-dimensional process $\{(X_n, d_n)\}_{n=0}^\infty$ can be also viewed as a Markov chain with respect to \mathcal{F}_n (the value of d_0 is irrelevant; furthermore, conditioning on d_n is irrelevant for (X_{n+1}, d_{n+1})).

The most natural case to examine first seems to be the choice $P\{\xi_n = -1\} = P\{\xi_n = 1\} = 1/2$ (or $P\{\xi_n = 1\} = P\{\xi_n = 3\} = 1/2$). Here, however, we will consider the rather easier case

$$P\{\xi_n = 1\} = P\{\xi_n = 3\} = P\{\xi_n = 5\} = P\{\xi_n = 7\} = \frac{1}{4}. \quad (2.5)$$

To begin our analysis, let us observe that for any positive odd integral value of X_{n-1} we have

$$\{3X_{n-1} + 1, 3X_{n-1} + 3, 3X_{n-1} + 5, 3X_{n-1} + 7\} \equiv \{0, 2, 4, 6\} \pmod{8}. \quad (2.6)$$

Therefore, due to (2.5) and the independence of X_{n-1} and ξ_n we have

$$P\{3X_{n-1} + \xi_n \equiv k \pmod{8}\} = \frac{1}{4} \quad \text{for } k = 0, 2, 4, 6. \quad (2.7)$$

The above formula motivates us to set

$$m_n := 3 \wedge d_n, \quad n \geq 1 \quad (2.8)$$

(as usual, $a \wedge b$ denotes the minimum of a and b), where d_n is the random exponent appearing in (2.1). Then, (2.1), (2.7), the Markov property of (X_n, d_n) , and the independence of X_{n-1} and ξ_n imply

$$P\{m_n = 1 \mid \mathcal{F}_{n-1}\} = P\{m_n = 1 \mid X_{n-1}\} = P\{3X_{n-1} + \xi_n \equiv 2 \pmod{4} \mid X_{n-1}\} = \frac{1}{2} \quad (2.9)$$

for all $n \geq 1$. Likewise,

$$P\{m_n = 2 \mid \mathcal{F}_{n-1}\} = P\{m_n = 3 \mid \mathcal{F}_{n-1}\} = \frac{1}{4}, \quad n \geq 1. \quad (2.10)$$

Formulas (2.9) and (2.10) tell us that m_n and \mathcal{F}_{n-1} are independent for every $n \geq 1$. In particular (since m_n is \mathcal{F}_n -measurable for all $n \geq 1$) we have that $\{m_n\}_{n=1}^\infty$ is a sequence of i.i.d. random variables with

$$P\{m_n = 1\} = \frac{1}{2}, \quad P\{m_n = 2\} = P\{m_n = 3\} = \frac{1}{4}. \quad (2.11)$$

Proposition 1. Let $\{X_n\}_{n=0}^\infty$ be the odd-integer-valued Markov chain introduced in (2.1), where $\{\xi_n\}_{n=1}^\infty$ is a sequence of i.i.d. random variables whose common distribution is given by (2.5). Then,

$$\limsup X_n = \infty \quad \text{a.s.} \quad (2.12)$$

for any initial value $X_0 = x$ (as usual, “a.s.” stands for “almost surely”, i.e. “with probability 1”).

Proof. Fix a constant $K > 0$ and then pick an integer $k \geq 1$ so that

$$\left(\frac{3}{2}\right)^k > K. \quad (2.13)$$

Since $\{m_n\}_{n=1}^\infty$ is a sequence of i.i.d. random variables whose common distribution is given by (2.11), an immediate consequence of the 2nd Borel-Cantelli Lemma is that

$$P\{m_n = m_{n+1} = \cdots = m_{n+k-1} = 1 \text{ i.o.}\} = 1 \quad (2.14)$$

(“i.o.” stands for “infinitely often”, i.e. for infinitely many values of n). But, if $m_n = m_{n+1} = \cdots = m_{n+k-1} = 1$, then, by (2.1), (2.8), (2.13), and the fact that $(X_0 = x \geq 1 \text{ and}) X_n, \xi_n \geq 1$ for every $n \geq 1$, we must also have that

$$X_{n+k} > \left(\frac{3}{2}\right)^k X_n > K. \quad (2.15)$$

Therefore, (2.14) implies that

$$P\{X_{n+k} > K \text{ i.o.}\} = 1. \quad (2.16)$$

From formula (2.16) we get

$$\limsup X_n \geq K \quad \text{a.s.}$$

and since K is arbitrary, the proposition follows from the fact that

$$\{\limsup X_n = \infty\} = \bigcap_{K=1}^{\infty} \{\limsup X_n \geq K\}.$$

■

Remark 1. In the case $P\{\xi_n = -1\} = P\{\xi_n = 1\} = 1/2$ things are quite different since now 1 is an absorbing (or trapping) state, i.e. if $X_n = 1$ for some n , then $X_{n+k} = 1$ for all $k \geq 0$. Hence, (2.15) and, consequently, (2.12)

are not valid. In fact, since for any initial state x there is always a positive probability that $X_n = 1$ for some n , it follows that $P\{\limsup X_n = \infty\} < 1$. A natural **open question** here, in the spirit of the Collatz Problem, is whether $P\{\limsup X_n = \infty\} = 0$.

Let us now continue the analysis of the case (2.5). Formula (2.1) can be written as

$$X_n = \frac{3}{2^{d_n}} \left(1 + \frac{\xi_n}{3X_{n-1}} \right) X_{n-1}. \quad (2.17)$$

Thus, in view of (2.5) and (2.8) we have

$$1 \leq X_n \leq \frac{3}{2^{m_n}} \left(1 + \frac{7}{3X_{n-1}} \right) X_{n-1}. \quad (2.18)$$

Due to the multiplicative form of the formulas it is convenient to set

$$Y_n := \ln X_n, \quad n \geq 0. \quad (2.19)$$

Then, inequality (2.18) is equivalent to

$$0 \leq Y_n \leq Y_{n-1} + \ln \left(1 + \frac{7}{3} e^{-Y_{n-1}} \right) + \ln 3 - m_n \ln 2. \quad (2.20)$$

By (2.11) we have $E[m_n] = 7/4$ (and $V[m_n] = 11/16$). Thus

$$E[\ln 3 - m_n \ln 2] = \ln 3 - \frac{7 \ln 2}{4} = -\frac{1}{4} \ln \left(\frac{128}{81} \right) \simeq -0.1144. \quad (2.21)$$

Now, let us fix an ε such that $0 < \varepsilon < -E[\ln 3 - m_n \ln 2]$. Then, it is easy to see that there is an $M > 1$ such that $\ln(1 + (7/3)e^{-y}) \leq \varepsilon$ for all $y \geq M$. For example, if $\varepsilon = 1/10$, then it suffices to take $M = \ln 23$. With such values of ε and M , formula (2.20) implies

$$0 \leq Y_n \leq Y_{n-1} + \varepsilon + \ln 3 - m_n \ln 2, \quad \text{if } Y_{n-1} \geq M, \quad (2.22)$$

and

$$0 \leq Y_n \leq Y_{n-1} + \ln 5, \quad \text{if } Y_{n-1} < M. \quad (2.23)$$

For notational convenience we prefer to write formula (2.22) in the form

$$0 \leq Y_n \leq Y_{n-1} + W_n, \quad \text{if } Y_{n-1} \geq M, \quad (2.24)$$

where

$$W_n := \varepsilon + \ln 3 - m_n \ln 2, \quad n \geq 0. \quad (2.25)$$

From the properties of $\{m_n\}_{n=1}^\infty$ it follows immediately that $\{W_n\}_{n=1}^\infty$ is a sequence of i.i.d. random variables with

$$\mu := E[W_n] = \varepsilon + E[\ln 3 - m_n \ln 2] < 0 \quad (2.26)$$

and, furthermore, that Y_{n-1} and W_n are independent for every $n \geq 1$. The following proposition is in the spirit of the Collatz Conjecture.

Proposition 2. Let $\{X_n\}_{n=0}^\infty$ be the Markov chain of Proposition 1. Then, the state 1 is positive recurrent [1]. In particular

$$P\{X_n = 1 \text{ i.o.}\} = 1 \quad (2.27)$$

for any initial value $X_0 = x$.

Proof. For an ε and an M as above let

$$N_1 := \inf\{n \geq 0 : Y_n \geq M\} \quad \text{and} \quad D_1 := \inf\{n > N_1 : Y_n < M\} \quad (2.28)$$

(if $Y_0 = \ln x \geq M$, then $N_1 = 0$). Notice that N_1 and D_1 are stopping times of the Markov chain $\{Y_n\}_{n=0}^\infty$ and, hence, of the filtration $\{\mathcal{F}_n\}_{n=1}^\infty$. By Proposition 1 we have that

$$N_1 < \infty \text{ a.s.} \quad (2.29)$$

Actually, much more is true. Since for any fixed $k \geq 1$ we have that

$$P\{m_n = m_{n+1} = \cdots = m_{n+k-1} = 1\} > 0,$$

it follows that there is an $\alpha_0 > 0$ such that

$$E[e^{\alpha N_1}] < \infty \quad \text{for all } \alpha < \alpha_0. \quad (2.30)$$

In particular,

$$E[N_1] < \infty. \quad (2.31)$$

Suppose now that $\omega \in \{D_1 = \infty\}$. Then, for such ω 's we must have $Y_{N_1+n} \geq M$ for all $n \geq 0$ and, hence, formula (2.24) becomes

$$Y_{N_1+n+1} - Y_{N_1+n} \leq W_{N_1+n+1}, \quad \text{for all } n \geq 0$$

or

$$Y_{N_1+n} - Y_{N_1} \leq \sum_{j=1}^n W_{N_1+j}, \quad \text{for all } n \geq 1. \quad (2.32)$$

However, $\{W_{N_1+j}\}_{j=1}^\infty$ is a sequence of i.i.d. random variables whose common distribution is that of W_n [1]. In particular, the common expectation of

W_{N_1+j} , $j \geq 1$, is strictly negative. As a consequence of these facts we have that the event of (2.32) has probability 0 and, furthermore, there is a $\beta_0 > 0$ such that

$$E[e^{\beta D_1}] < \infty \quad \text{for all } \beta < \beta_0. \quad (2.33)$$

In particular,

$$E[D_1] < \infty \quad (2.34)$$

and

$$D_1 < \infty \text{ a.s.} \quad (2.35)$$

We can then introduce the stopping times

$$N_k := \inf\{n > D_{k-1} : Y_n \geq M\}, \quad D_k := \inf\{n > N_k : Y_n < M\}, \quad k \geq 2. \quad (2.36)$$

As in the case of N_1 and D_1 , we again have that, there are $\alpha_0 > 0$ and $\beta_0 > 0$ independent of k such that, for all $k \geq 2$ we have

$$E[e^{\alpha N_k}] < \infty, \quad E[e^{\beta D_k}] < \infty \quad \text{for every } \alpha < \alpha_0, \beta < \beta_0. \quad (2.37)$$

In particular,

$$E[N_k] < \infty, \quad E[D_k] < \infty \quad (2.38)$$

and

$$N_k < \infty, \quad D_k < \infty \text{ a.s.} \quad (2.39)$$

Let \mathcal{O}_M denote the set of odd positive integers which are less than e^M (definitely $1 \in \mathcal{O}_M$). Then, the above analysis implies

$$P\{X_n \in \mathcal{O}_M \text{ i.o.}\} = 1. \quad (2.40)$$

Since \mathcal{O}_M is a finite set, some state r in \mathcal{O}_M must be recurrent; actually positive recurrent due to (2.38) [1]. But, then, since there is a nonzero probability for the Markov chain X_n to go from any $r \in \mathcal{O}_M$ to 1, it follows that 1 is positive recurrent [1]. \blacksquare

The next proposition generalizes Proposition 2.

Proposition 3. Let $\{X_n\}_{n=0}^\infty$ be the Markov chain of Propositions 1 and 2. Then, all states in \mathbb{N}_{odd} are positive recurrent.

Proof. By Proposition 2 we can assume without loss of generality that $X_0 = 1$. Then, we need to show that all states in \mathbb{N}_{odd} can be reached by $\{X_n\}_{n=0}^\infty$ with nonzero probability.

Let m be the smallest odd integer which cannot be reached, namely $P\{X_n = m\} = 0$ for all $n \geq 1$. By Proposition 2 we have that $m \geq 3$. Therefore, we should have one of the following three possibilities:

$$m = 6k + 3 \quad \text{or} \quad m = 6k + 5 \quad \text{or} \quad m = 6k + 7 \quad \text{for some } k \geq 0.$$

(i) Suppose $m = 6k + 3$ for some $k \geq 0$. Then, $4k + 1 < m$ and since $4k + 1$ is odd, we must have that $P\{X_n = 4k + 1\} > 0$ for some n . But, then, since

$$3(4k + 1) + 3 = 12k + 6 = 2(6k + 3) = 2m,$$

we should have by (2.1) that $P\{X_{n+1} = m\} > 0$, a contradiction.

(ii) Next, suppose $m = 6k + 5$ for some $k \geq 0$. Then, $4k + 3 < m$ and since $4k + 3$ is odd, we must have that $P\{X_n = 4k + 3\} > 0$ for some n . But, then, since

$$3(4k + 3) + 1 = 12k + 10 = 2(6k + 5) = 2m,$$

we should have by (2.1) that $P\{X_{n+1} = m\} > 0$, a contradiction.

(iii) Finally, suppose $m = 6k + 7$ for some $k \geq 0$. Then, $4k + 3 < m$ and since $4k + 3$ is odd, we must have that $P\{X_n = 4k + 3\} > 0$ for some n . But, then, since

$$3(4k + 3) + 5 = 12k + 14 = 2(6k + 7) = 2m,$$

we should have by (2.1) that $P\{X_{n+1} = m\} > 0$, again a contradiction.

Therefore, in all three possibilities for m we have reached a contradiction. It follows that such an m cannot exist. ■

Remark 2. A side result of Proposition 3 is that the Markov chain $\{X_n\}_{n=0}^{\infty}$ is irreducible [1]. In the case where ξ_n takes the values 1, 3, and 5 only with positive probabilities, given $X_0 = 1$, one can use the idea of the proof of Proposition 3 in order to show that the associated Markov chain is irreducible. However, it is an **open question** whether Proposition 2 is valid in that case. As for the case where ξ_n takes only the values 1 and 3 with positive probabilities, given $X_0 = 1$, even the irreducibility is an **open question**.

Final Comments. Since $P\{X_{n+1} = 1 \mid X_n = 1\} = 1/2 > 0$, it follows from Proposition 3 that the Markov chain $\{X_n\}_{n=0}^{\infty}$ is aperiodic [1]. Also, again by Proposition 3 we have that $\{X_n\}_{n=0}^{\infty}$ has a stationary distribution π [1]. Finally, the existence of π together with the aperiodicity (and the irreducibility mentioned in Remark 2) imply [1] that

$$P\{X_n = y \mid X_0 = x\} \rightarrow \pi(y) \quad \text{as } n \rightarrow \infty,$$

for every $x, y \in \mathbb{N}_{\text{odd}}$.

References

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